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Coherent states for isospectral oscillator Hamiltonians

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Abstract. Coherent states for a family of isospectral oscillator Hamiltonians are derived from a suitable choice of annihilation and creation operators. The Fock–Bargmann representation is also considered.

1. Introduction

Starting from the harmonic oscillator, a family of isospectral Hamiltonians has been obtained [1] using a generalization of the well known factorization method [2]. A connection has been made between this method and the supersymmetry transformation [3]. In this way, the harmonic oscillator and the isospectral family of Hamiltonians may be seen as supersymmetric (SUSY) partners [4]. This presentation of the factorization method has the advantage of giving a global view of the different Hamiltonians with which we are dealing. It also provides a way of constructing an orthonormal set of basis vectors for the isospectral Hamiltonians departing from the usual harmonic oscillator basis.

In recent years, however, the necessity for studying alternative sets of basis vectors (not necessarily orthonormals) has been realized. The most important of these is the set of coherent states [5, 6]. For the harmonic oscillator, they are very well known and have proved to be useful in many branches of physics [5]. For the isospectral Hamiltonians derived in [1], there has been no such study. The goal of this paper is to fill this gap by making such a study. It will be shown that this can be achieved by appropriately defining new annihilation and creation operators.

The plan of the paper is as follows. In section 2, we give the family of isospectral Hamiltonians, their eigenstates and a pair of associated annihilation and creation operators. In section 3, we construct the coherent states and analyse their dynamical evolution and their overcompleteness. The harmonic oscillator limit is considered in section 4. It leads to an explicit calculation of the uncertainty relation. Finally, the Fock–Bargmann representation is studied in section 5.

2. The factorization method and isospectral oscillators

We will start with the SUSY approach to the factorization method. Let us consider the following operators:

$$a = \frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x} + U'(x)\right) \qquad a^{+} = \frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}x} + U'(x)\right) \tag{2.1}$$

where $U'(x)$ is the derivative of $U(x)$, an arbitrary function of x. We get a SUSY Hamiltonian [4]

$$H_{\mathrm{SUSY}} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} = \begin{pmatrix} a^+a & 0 \\ 0 & aa^+ \end{pmatrix}. \tag{2.2}$$

$H_\pm$ are the SUSY partners given explicitly as

$$H_\pm = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_\pm(x) \tag{2.3}$$

with

$$V_\pm(x) = \tfrac{1}{2}\left[(U')^2 \mp U''\right]. \tag{2.4}$$

It is well known that the ground state of H_+ has zero energy provided that the ground-state wavefunction

$$\psi_0(x) = N_0 \exp\bigl(-U(x)\bigr) \tag{2.5}$$

is square-integrable. Moreover, the eigenfunctions and eigenvalues of $H_\pm$ are related [3]. This means that starting, for example, with H_+ as a known solvable problem, we construct a new Hamiltonian H_- which is also solvable.

Evidently, for some simple cases such as the harmonic oscillator, the previous result is trivial. Nevertheless, it has been shown that this method produces a large class of non-trivial solvable potentials if, in addition, they verify some specific properties [3].

Here we want to insist on another point resulting from the factorization: the non-unicity of the definition of a and a^+. This was first observed by Mielnik in the harmonic oscillator case and has been formalized for arbitrary potentials of the form (2.4) by Nieto [4] in the SUSY context. Indeed, if we look for operators b and b^+ of the form

$$b = \frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x} + \beta(x)\right) \qquad b^+ = \frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}x} + \beta(x)\right) \tag{2.6}$$

such that bb^+ coincides with H_-, one gets a Riccati equation for $\beta(x)$

$$\beta' + \beta^2 = U'' + (U')^2 \tag{2.7}$$

the general solution of which is

$$\beta(x) = U'(x) + \phi_\lambda(x) \tag{2.8}$$

where

$$\phi_\lambda(x) = \frac{\mathrm{e}^{-2U(x)}}{\lambda + \int_0^x \mathrm{e}^{-2U(y)}\,\mathrm{d}y} \qquad \lambda \in \mathbb{R}. \tag{2.9}$$

The SUSY Hamiltonian corresponding to the operators (2.6) now takes the form

$$H_{\lambda,\mathrm{SUSY}} = \begin{pmatrix} H_{\lambda,+} & 0 \\ 0 & H_{\lambda,-} \end{pmatrix} = \begin{pmatrix} b^+b & 0 \\ 0 & bb^+ \end{pmatrix} \tag{2.10}$$

where $H_{\lambda,-} = H_-$, as required, but $H_{\lambda,+}$ is

$$H_{\lambda,+} = H_+ - \phi'_\lambda(x). \tag{2.11}$$

An interpretation of this Hamiltonian in terms of the Gel'fand–Levitan method may be found in Nieto [4]. It is essentially different from H_+ except in the limit $|\lambda| \to \infty$.

The particular case we are interested in discussing further is that of the harmonic oscillator. Indeed, let us take $U(x) = \frac{1}{2}x^2$ so that the operators (2.1) become, evidently, the annihilation and creation operators for the harmonic oscillator Hamiltonian

$$H = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}x^2 \tag{2.12}$$

and

$$a^+a = H - \tfrac{1}{2} \qquad aa^+ = H + \tfrac{1}{2}. \tag{2.13}$$

The factorization in terms of the operators b and b^+ then leads to

$$b^+b = H_\lambda - \tfrac{1}{2} \qquad bb^+ = H + \tfrac{1}{2} \tag{2.14}$$

where

$$H_\lambda = H_{\lambda,+} + \tfrac{1}{2} = H - \phi'_\lambda(x) = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_\lambda(x) \tag{2.15}$$

with

$$V_\lambda(x) = \frac{x^2}{2} - \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{e}^{-x^2}}{\lambda + \int_0^x \mathrm{e}^{-y^2}\,\mathrm{d}y}\right]. \tag{2.16}$$

To have an idea of the behaviour of $V_\lambda(x)$, we have plotted it for different values of λ in figure 1. If we want $V_\lambda(x)$ to be non-singular, we must have $|\lambda| > \sqrt{\pi}/2$ [1]. Notice that for $|\lambda| \to \infty$, $H_\lambda \to H$.

Due to the relation

$$H_\lambda b^+ = b^+(H+1) \tag{2.17}$$

it is easy to see that the states

$$|\theta_n\rangle = b^+|\psi_{n-1}\rangle/\sqrt{n} \qquad n = 1, 2, \ldots \tag{2.18}$$

are normalized orthogonal eigenvectors of H_λ with eigenvalues $E_n = n + \frac{1}{2}$. The states

$$|\psi_n\rangle = \frac{(a^+)^n|\psi_0\rangle}{\sqrt{n!}} \tag{2.19}$$

are the normalized eigenstates of the harmonic oscillator. Notice that the set $\{|\theta_n\rangle, n = 1, 2, \ldots\}$ is not complete. The missing vector $|\theta_0\rangle$, orthogonal to the others, verifies $b|\theta_0\rangle = 0$ and is given in the coordinate representation by

$$\theta_0(x) = C_0\mathrm{e}^{-x^2/2}\exp\left(-\int_0^x \phi_\lambda(y)\,\mathrm{d}y\right) = \frac{C_0\mathrm{e}^{-x^2/2}}{\lambda + \int_0^x \mathrm{e}^{-y^2}\,\mathrm{d}y}. \tag{2.20}$$

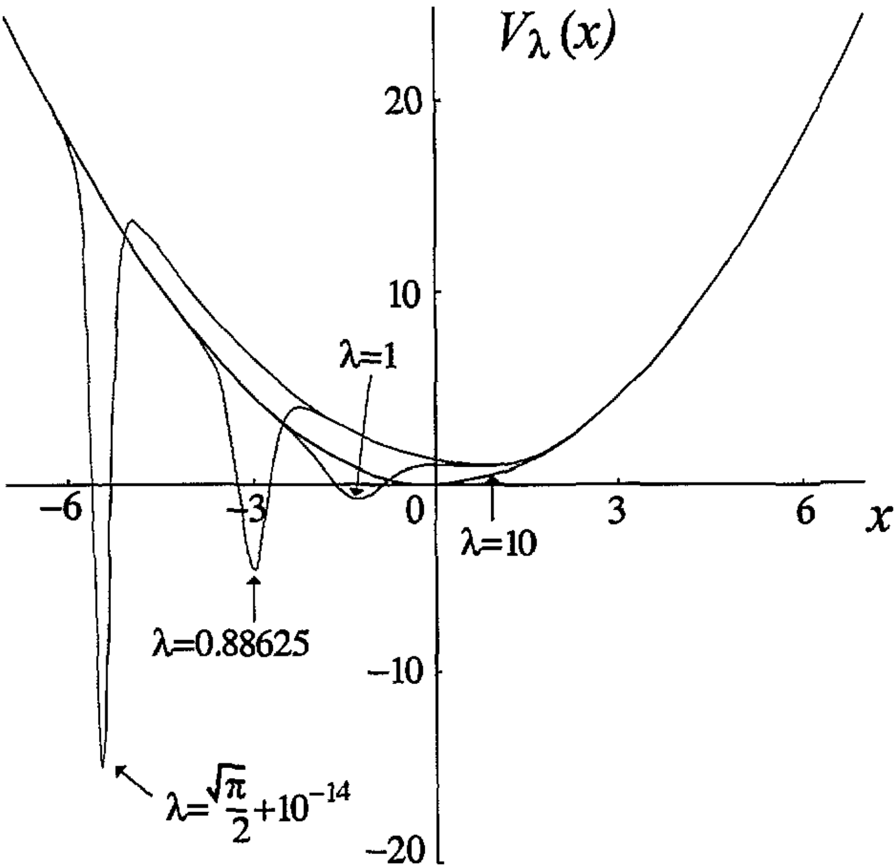

Figure 1. The isospectral oscillator potentials $V_\lambda(x)$.

It is an eigenvector of H_λ with eigenvalue $\frac{1}{2}$. The set $\{|\theta_n\rangle, n = 0, 1, 2, \ldots\}$ is now complete in $L^2(\mathbb{R})$, therefore H_λ is a Hamiltonian with a spectrum equal to that of the harmonic oscillator.

We are interested now in identifying the annihilation and creation operators for the Hamiltonian H_λ. Let us denote them A and A^+. It can easily be seen [1] that they are simply

$$A = b^+ab \qquad A^+ = b^+a^+b. \tag{2.21}$$

The action of the whole set of operators acting on the states of the harmonic oscillator $\{|\psi_n\rangle\}$, and on the generalized oscillator states $\{|\theta_n\rangle\}$, can easily be visualized in figure 2 and is summarized by the following formulae:

$$b|\theta_n\rangle = \sqrt{n}|\psi_{n-1}\rangle \qquad b^+|\psi_n\rangle = \sqrt{n+1}|\theta_{n+1}\rangle. \tag{2.22}$$

3. Coherent states

It is well known that there are several non-equivalent definitions of coherent states [5,6]. One of the possibilities is to look for the eigenstates of an annihilation operator. We have

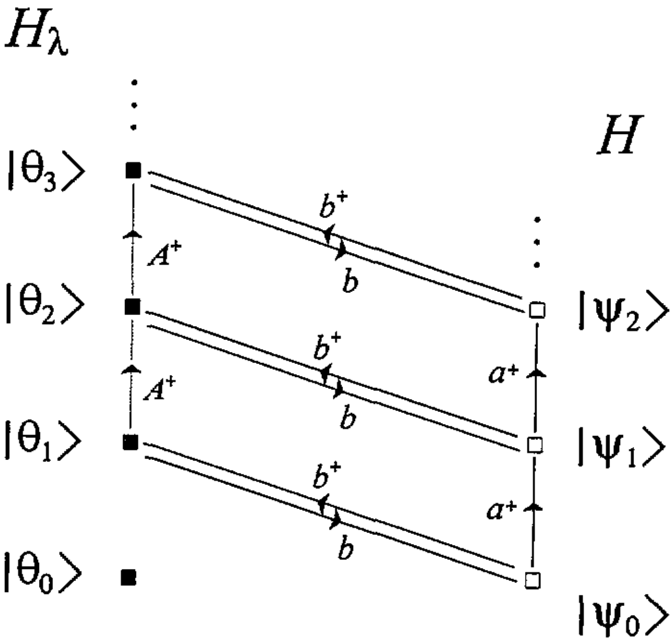

Figure 2. The action of various operators on the basis vectors.

seen that A, given in (2.21), is such an operator and will be used to derive the coherent states associated with our family of Hamiltonians. In other words, the state $|z\rangle$ for which we are looking must verify

$$A|z\rangle = z|z\rangle. \tag{3.1}$$

Here, the connection with a possible group-theoretical approach cannot be applied because, in opposition to the harmonic oscillator, the operators A and A^+ do not give a closed algebra. Although $[A, A^+] \neq I$, we can find an operator B such that

$$[B, A^+] = I \qquad [A, B^+] = I. \tag{3.2}$$

This is given by

$$B = b^+ a \frac{1}{N(1+N)} b. \tag{3.3}$$

This fact can easily be proved by using the equality

$$f(H_\lambda) b^+ = b^+ f(H+1) \tag{3.4}$$

for an arbitrary function f. It is then easy to rewrite $|z\rangle$ (up to normalization) as

$$|z\rangle = e^{zB^+} |\theta_1\rangle \tag{3.5}$$

but this does not correspond to the action of a unitary representation of either of the two algebras in (3.2).

Let us now compute $|z\rangle$ explicitly using (3.1). We take

$$|z\rangle = \sum_{n=0}^{\infty} a_n |\theta_n\rangle \tag{3.6}$$

so that

$$A|z\rangle = \sum_{m=1}^{\infty} a_{m+1}\sqrt{m^2(m+1)}|\theta_m\rangle = z\left(a_0|\theta_0\rangle + \sum_{m=1}^{\infty} a_m|\theta_m\rangle\right). \tag{3.7}$$

Then, all the coefficients are given by

$$a_0 = 0 \qquad a_{m+1} = \frac{z^m}{m!\sqrt{(m+1)!}}a_1 \qquad m = 0, 1, 2, \dots . \tag{3.8}$$

The dependence on a_1 is removed by imposing the normalization and choosing it to be real. We find

$$a_1 = [{}_0F_2(1, 2; |z|^2)]^{-1/2} \tag{3.9}$$

where ${}_0F_2(1, 2; |z|^2)$ is a generalized hypergeometric function defined as [7]

$${}_0F_2(\alpha, \beta; x) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+n)\Gamma(\beta+n)}\frac{x^n}{n!}. \tag{3.10}$$

From this expression, it is clear that ${}_0F_2(1, 2; |z|^2)$ is a positive definite function on $\mathbb{C}$ with radial symmetry. The final expression for the coherent state is

$$|z\rangle = \frac{1}{\sqrt{{}_0F_2(1, 2; |z|^2)}} \sum_{n=0}^{\infty} \frac{z^n}{n!\sqrt{(n+1)!}}|\theta_{n+1}\rangle. \tag{3.11}$$

We see that $z = 0$ is a doubly degenerate eigenvalue for A. The corresponding eigenvectors are $|0\rangle \equiv |\theta_1\rangle$ and $|\theta_0\rangle$.

Let us analyse now the completeness (indeed, the overcompleteness) of the set $\{|\theta_0\rangle, |z\rangle; z \in \mathbb{C}\}$. We are looking for the resolution of the identity on $\mathcal{H}$. As $|\theta_0\rangle$ is 'isolated' from the other states $\{|z\rangle\}$, this resolution of the identity must take the form

$$I_{\mathcal{H}} = |\theta_0\rangle\langle\theta_0| + \int |z\rangle\langle z|\,\mathrm{d}\mu(z) \tag{3.12}$$

where the measure $\mathrm{d}\mu(z)$ has to be determined. As there is no group structure involved in our treatment, it does not make sense to look for an invariant measure. If we suppose that $\mathrm{d}\mu(z)$ depends only on $|z|$, it can be determined as in [8]. Indeed, let us take

$$\mathrm{d}\mu(z) = {}_0F_2(1, 2; r^2)h(r^2)r\,\mathrm{d}r\,\mathrm{d}\varphi \qquad z = r\mathrm{e}^{\mathrm{i}\varphi}. \tag{3.13}$$

Then, performing the integral in the angular variable φ, we get

$$I_{\mathcal{H}} = |\theta_0\rangle\langle\theta_0| + \sum_{n=1}^{\infty} |\theta_n\rangle\langle\theta_n|\left[\frac{\pi}{((n-1)!)^2 n!}\int_0^{\infty} h(x)x^{n-1}\,\mathrm{d}x\right]. \tag{3.14}$$

In order to recover the resolution of the identity in terms of the basis $\{|\theta_n\rangle\}$, we must have

$$\int_0^\infty h(x)x^{n-1}\,\mathrm{d}x = \frac{(\Gamma(n))^2\Gamma(n+1)}{\pi} \qquad n = 1, 2, \ldots . \tag{3.15}$$

Then, it is clear that the function $h(x)$ we are looking for is the inverse Mellin transform of $(\Gamma(s))^2\Gamma(s+1)/\pi$. It is evaluated in appendix A. The result is

$$\begin{aligned} h(x) = &-\frac{3\gamma}{\pi} + \frac{x(\log x)^2}{2\pi}\,{}_0F_2(2,2;-x) - \frac{\log x}{\pi}\,{}_0F_2(1,1;-x) \\ &- \frac{3x\log x}{\pi}\sum_{n=0}^{\infty}\gamma_{n+1}\frac{(-x)^n}{((n+1)!)^2 n!} - \frac{3x}{\pi}\sum_{n=0}^{\infty}\gamma_{n+1}\frac{(-x)^n}{((n+1)!)^3} \\ &+ \frac{9x}{\pi}\sum_{n=0}^{\infty}\gamma_{n+1}^2\frac{(-x)^n}{((n+1)!)^2 n!} + \frac{3x}{2\pi}\sum_{n=0}^{\infty}\left(\frac{\pi^2}{6} + \sum_{k=1}^{n+1}\frac{1}{k^2}\right)\frac{(-x)^n}{((n+1)!)^2 n!} \end{aligned} \tag{3.16}$$

where γ is Euler's constant and $\gamma_n = \psi(1+n) = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \gamma$. The previous result, along with (3.13), gives us a positive measure for the case we are considering. Notice that, although $h(r^2)$ is singular at the origin (as is clear from equation (3.16)), the measure is not.

There are two main consequences arising from the former result. First, we can express any coherent state $|z'\rangle$ in terms of the others

$$|z'\rangle = \int |z\rangle\langle z|z'\rangle\,\mathrm{d}\mu(z) \qquad |z'\rangle \neq |\theta_0\rangle. \tag{3.17}$$

The kernel $\langle z|z'\rangle$ is easy to evaluate from (3.11)

$$\langle z|z'\rangle = \frac{{}_0F_2(1,2;\bar{z}z')}{\sqrt{{}_0F_2(1,2;|z|^2)\,{}_0F_2(1,2;|z'|^2)}} \tag{3.18}$$

and it is trivial to prove that this is a reproducing kernel, that is

$$\int \langle z'|z\rangle\langle z|z''\rangle\,\mathrm{d}\mu(z) = \langle z'|z''\rangle \qquad |z'\rangle, |z''\rangle \neq |\theta_0\rangle. \tag{3.19}$$

Second, an arbitrary element of the Hilbert space $\mathcal{H}$, let us call it $|g\rangle$, can be written in terms of the coherent states

$$|g\rangle = g_0|\theta_0\rangle + \int \tilde{g}(z,\bar{z})|z\rangle\,\mathrm{d}\mu(z) \tag{3.20}$$

where $g_0 = \langle\theta_0|g\rangle$ and

$$\tilde{g}(z,\bar{z}) = \langle z|g\rangle = \frac{1}{\sqrt{{}_0F_2(1,2;|z|^2)}}\sum_{n=0}^{\infty}\frac{\bar{z}^n}{n!\sqrt{(n+1)!}}\langle\theta_{n+1}|g\rangle. \tag{3.21}$$

The function $\tilde{g}(z,\bar{z})$ and the number g_0 determine completely the state $|g\rangle \in \mathcal{H}$.

Let us now consider the dynamical evolution of the coherent states, which is, indeed, quite simple, due to the fact that the eigenvalues of H_λ are the same as those of the harmonic oscillator. More precisely,

$$
\begin{aligned}
U(t)|z\rangle &= \frac{1}{\sqrt{{}_0F_2(1,2,|z|^2)}}\sum_{n=0}^{\infty}\frac{z^n}{n!\sqrt{(n+1)!}}\mathrm{e}^{-\mathrm{i}tH_\lambda}|\theta_{n+1}\rangle \\
&= \frac{1}{\sqrt{{}_0F_2(1,2,|z|^2)}}\sum_{n=0}^{\infty}\frac{z^n\mathrm{e}^{-\mathrm{i}t(n+3/2)}}{n!\sqrt{(n+1)!}}|\theta_{n+1}\rangle \\
&= \mathrm{e}^{-\mathrm{i}3t/2}|z(t)\rangle \qquad z(t)=\mathrm{e}^{-\mathrm{i}t}z.
\end{aligned}
\tag{3.22}
$$

We can also compute the expected value of the Hamiltonian H_λ in a coherent state:

$$
\begin{aligned}
\langle z|H_\lambda|z\rangle &= \frac{1}{{}_0F_2(1,2;|z|^2)}\sum_{m,n=0}^{\infty}\frac{\bar{z}^m z^n}{m!n!\sqrt{(m+1)!(n+1)!}}\langle\theta_{m+1}|H_\lambda|\theta_{n+1}\rangle \\
&= \frac{1}{{}_0F_2(1,2;|z|^2)}\sum_{n=0}^{\infty}\frac{|z|^{2n}(n+3/2)}{(n!)^2(n+1)!} = \frac{{}_0F_2(1,1;|z|^2)}{{}_0F_2(1,2;|z|^2)}+\frac{1}{2}.
\end{aligned}
\tag{3.23}
$$

4. The harmonic oscillator limit

It is clear from (2.15) and (2.16) that the uniparametric family H_λ tends to the harmonic oscillator Hamiltonian in the limit $|\lambda| \to \infty$. Let us now consider this limit in detail to see if there is a relationship between the coherent states we have computed and the harmonic oscillator ones. In the limit, $\beta(x) \to x$, and therefore $b \to a$ and $b^+ \to a^+$. Then, we get

$$
|\theta_n\rangle \to \frac{a^+|\psi_{n-1}\rangle}{\sqrt{n}} = |\psi_n\rangle \qquad n = 1, 2, \ldots. \tag{4.1}
$$

We also have $|\theta_0\rangle \to |\psi_0\rangle$. This is not evident from (2.20), but notice that the normalization constant C_0 also depends on λ in a non-trivial way. Nevertheless, the correct limit comes from the definition of $|\theta_0\rangle$; that is, $b|\theta_0\rangle = 0$. The operator A goes to $A_0 = a^+a^2$. As a consequence, the coherent states given in (3.11) become

$$
|z\rangle_0 \equiv \lim_{|\lambda|\to\infty}|z\rangle = \frac{1}{\sqrt{{}_0F_2(1,2;|z|^2)}}\sum_{n=0}^{\infty}\frac{z^n}{n!\sqrt{(n+1)!}}|\psi_{n+1}\rangle \tag{4.2}
$$

which are not the habitual coherent states. They have an identity resolution similar to (3.12), indeed,

$$
I_{\mathcal{H}} = |\psi_0\rangle\langle\psi_0| + \int |z\rangle_{00}\langle z|\,\mathrm{d}\mu(z) \tag{4.3}
$$

where the measure $\mathrm{d}\mu(z)$ is the one computed in section 3, given by (3.13) and (3.16). The time evolution of $|z\rangle_0$ and the expected value ${}_0\langle z|H|z\rangle_0$ are the same as those found in (3.22) and (3.23).

For the state $|z\rangle$, it is very difficult to compute the expectation values of the position and momentum operators, but for the state $|z\rangle_0$ the problem can be solved easily. Let us consider the state $|z\rangle_0$ given in (4.2). It is very well known that the position and momentum operators can be written in terms of the creation and annihilation operators a^+, a

$$\hat{x} = \frac{1}{\sqrt{2}}(a^+ + a) \qquad \hat{p} = \frac{\mathrm{i}}{\sqrt{2}}(a^+ - a). \tag{4.4}$$

Using the action of a^+ and a on the state $|\psi_n\rangle$, we have

$$\begin{aligned}
{}_0\langle z|\hat{x}|z\rangle_0 &= \frac{1}{{}_0F_2(1,2;|z|^2)} \sum_{m,n=0}^{\infty} \frac{\bar{z}^m z^n}{m!n!\sqrt{(m+1)!(n+1)!}} \langle\psi_{m+1}|\hat{x}|\psi_{n+1}\rangle \\
&= \frac{1}{\sqrt{2}{}_0F_2(1,2;|z|^2)} \sum_{m,n=0}^{\infty} \frac{\bar{z}^m z^n \left(\sqrt{n+2}\delta_{m+1,n+2} + \sqrt{n+1}\delta_{m+1,n}\right)}{m!n!\sqrt{(m+1)!(n+1)!}} \\
&= \frac{1}{\sqrt{2}{}_0F_2(1,2;|z|^2)} \left[\bar{z}\sum_{n=0}^{\infty} \frac{|z|^{2n}}{((n+1)!)^2 n!} + z\sum_{m=0}^{\infty} \frac{|z|^{2m}}{(m!)((m+1)!)^2}\right] \\
&= \frac{\bar{z}+z}{\sqrt{2}} \frac{{}_0F_2(2,2;|z|^2)}{{}_0F_2(1,2;|z|^2)}.
\end{aligned} \tag{4.5}$$

In a similar way, we obtain

$${}_0\langle z|\hat{x}^2|z\rangle_0 = \frac{1}{2{}_0F_2(1,2;|z|^2)} \left(3{}_0F_2(1,2;|z|^2) + \frac{(\bar{z}+z)^2}{2}{}_0F_2(2,3;|z|^2)\right). \tag{4.6}$$

Therefore,

$$\begin{aligned}
(\Delta\hat{x})^2 &= {}_0\langle z|\hat{x}^2|z\rangle_0 - ({}_0\langle z|\hat{x}|z\rangle_0)^2 \\
&= \frac{1}{2{}_0F_2(1,2;|z|^2)^2}\Big(3{}_0F_2(1,2;|z|^2)^2 \\
&\quad + \frac{(\bar{z}+z)^2}{2}\left[{}_0F_2(1,2;|z|^2){}_0F_2(2,3;|z|^2) - 2{}_0F_2(2,2;|z|^2)^2\right]\Big).
\end{aligned} \tag{4.7}$$

For the momentum operator we have similar results

$${}_0\langle z|\hat{p}|z\rangle_0 = \frac{\mathrm{i}(\bar{z}-z)}{\sqrt{2}} \frac{{}_0F_2(2,2;|z|^2)}{{}_0F_2(1,2;|z|^2)} \tag{4.8}$$

$${}_0\langle z|\hat{p}^2|z\rangle_0 = \frac{1}{2{}_0F_2(1,2;|z|^2)} \left(3{}_0F_2(1,2;|z|^2) - \frac{(\bar{z}-z)^2}{2}{}_0F_2(2,3;|z|^2)\right) \tag{4.9}$$

$$\begin{aligned}
(\Delta\hat{p})^2 &= {}_0\langle z|\hat{p}^2|z\rangle_0 - ({}_0\langle z|\hat{p}|z\rangle_0)^2 \\
&= \frac{1}{2{}_0F_2(1,2;|z|^2)^2}\Big(3{}_0F_2(1,2;|z|^2)^2 \\
&\quad - \frac{(\bar{z}-z)^2}{2}\left[{}_0F_2(1,2;|z|^2){}_0F_2(2,3;|z|^2) - 2\,{}_0F_2(2,2;|z|^2)^2\right]\Big).
\end{aligned} \tag{4.10}$$

If we define the function

$$\varrho(|z|) = \frac{{}_0F_2(1,2;|z|^2)\,{}_0F_2(2,3;|z|^2) - 2\,{}_0F_2(2,2;|z|^2)^2}{{}_0F_2(1,2;|z|^2)^2} \tag{4.11}$$

we can write the uncertainty relation for this case as

$$(\Delta\hat{x})(\Delta\hat{p}) = \sqrt{\left(\tfrac{3}{2}\right)^2 + \tfrac{3}{2}|z|^2\varrho(|z|) + [(\mathrm{Re}\, z)(\mathrm{Im}\, z)\varrho(|z|)]^2}. \tag{4.12}$$

A plot of this function is shown in figure 3. It can be proved rigorously that $\frac{1}{2} \leqslant (\Delta\hat{x})(\Delta\hat{p}) \leqslant \frac{3}{2}$. Then, $(\Delta\hat{x})(\Delta\hat{p})$ is almost at its minimum, suggesting the possibility of using $|z\rangle_0$ as quasiclassical states in the same sense as the usual coherent states.

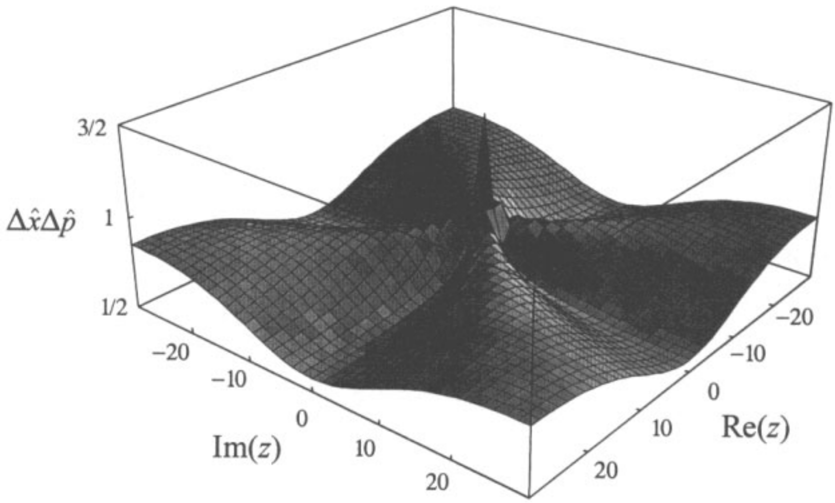

Figure 3. The uncertainty product $(\Delta\hat{x})(\Delta\hat{p})$ as a function of z.

5. The Fock–Bargmann representation

It is well known [6, 9] that for the harmonic oscillator it is possible to find a realization of the Hilbert space where any state vector is described by an entire function. The same is true for the coherent states associated with the Lie algebra su(1, 1) [8, 10]. Next, we will show that we can construct a similar realization for the problem under study. To achieve this, we can use the coherent states $|z\rangle$ or their limit $|z\rangle_0$. The result is going to be the same and, therefore, we will work with $|z\rangle$ given in (3.11).

Let us remember that the Hilbert space $\mathcal{H}$ is generated by the basis vectors $\{|\theta_0\rangle, |\theta_1\rangle, |\theta_2\rangle, \ldots\}$. We have already seen that the state $|\theta_0\rangle$ is isolated from the others in the sense that it is an atypical coherent state. Let us call $\mathcal{H}_0$ the one-dimensional subspace generated by $|\theta_0\rangle$ and $\mathcal{H}_1$ the Hilbert space generated by $\{|\theta_1\rangle, |\theta_2\rangle, \ldots\}$ so that $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. From now on, we are going to concentrate on $\mathcal{H}_1$. A vector $|g\rangle \in \mathcal{H}_1$ is characterized by $g_0 = 0$ and $\tilde{g}(z, \bar{z})$ given in (3.21).

A realization of $\mathcal{H}_1$ as a space $\mathcal{F}$ of entire analytic functions is obtained by taking

$$g(z) = \sum_{n=0}^{\infty} \frac{\langle\theta_{n+1}|g\rangle}{n!\sqrt{(n+1)!}} z^n. \tag{5.1}$$

This is related to $\tilde{g}(z, \bar{z})$ by

$$\tilde{g}(z, \bar{z}) = \langle z|g\rangle = \frac{g(\bar{z})}{\sqrt{{}_0F_2(1, 2; |z|^2)}} \tag{5.2}$$

and its norm is

$$\|g\|^2 = \langle g|g\rangle = \int |g(z)|^2 h(|z|^2)\, \mathrm{d}z \equiv \int |g(z)|^2\, \mathrm{d}\sigma(z) \tag{5.3}$$

due to definition (3.13) of the measure.

Let us study the functional space $\mathcal{F}$. From the relation (issued from the Schwarz inequality)

$$|g(z)| \leqslant \|g\| \sqrt{{}_0F_2(1, 2; |z|^2)} \qquad \forall g(z) \in \mathcal{F} \tag{5.4}$$

we can show that $g(z)$ is an entire function of order $\frac{2}{3}$ and type $\frac{3}{2}$ (see appendix B). Therefore, $\mathcal{F}$ is a subspace of the space of entire functions of growth $(\frac{2}{3}, \frac{3}{2})$ composed of functions of finite norm with respect to the measure $\mathrm{d}\sigma(z)$ (the usual coherent states are related to the Segal–Bargmann space of entire functions of growth $(\frac{1}{2}, 2)$). In particular, the entire function corresponding to a coherent state $|\alpha\rangle$ is

$$\alpha(z) = \sqrt{{}_0F_2(1, 2; |z|^2)}\langle\bar{z}|\alpha\rangle = \frac{1}{\sqrt{{}_0F_2(1, 2; |\alpha|^2)}} {}_0F_2(1, 2; \alpha z) \tag{5.5}$$

where we have used the kernel given in (3.18).

The functions $\theta_{n+1}(z)$ defined by

$$\theta_{n+1}(z) = \frac{z^n}{n!\sqrt{(n+1)!}} \qquad n = 0, 1, 2, \ldots \tag{5.6}$$

form an orthonormal basis of $\mathcal{F}$ so that the function $g(z)$ may be written

$$g(z) = \sum_{n=0}^{\infty} c_{n+1}\theta_{n+1}(z) \qquad c_{n+1} = \langle\theta_{n+1}|g\rangle. \tag{5.7}$$

Let us mention that the function

$$\delta(z, z') = \sum_{n=0}^{\infty} \theta_{n+1}(z)\bar{\theta}_{n+1}(z') = {}_0F_2(1, 2; z\bar{z}') \tag{5.8}$$

closely related to the reproducing kernel, plays the role of the delta function in the space $\mathcal{F}$ with respect to the measure $\mathrm{d}\sigma(z')$. This fact is very easy to prove

$$\int g(z')\,{}_0F_2(1, 2; z\bar{z}')\, \mathrm{d}\sigma(z') = \sqrt{{}_0F_2(1, 2, |z|^2)} \int \langle\bar{z}'|g\rangle\langle\bar{z}|\bar{z}'\rangle\, \mathrm{d}\mu(z') = g(z). \tag{5.9}$$

To summarize, we have proved that, to a given $|g\rangle \in \mathcal{H}_1$, we can associate a function $g(z) \in \mathcal{F}$ entirely of growth $(\frac{2}{3}, \frac{3}{2})$ and normalizable. On the other hand, it is obvious that for every $g(z) \in \mathcal{F}$ we can build a ket $|g\rangle \in \mathcal{H}_1$.

To finish, we need to know the abstract realization of the operators that act on $\mathcal{F}$ as a multiplication by z and as a derivation $\partial/\partial z$. Let us consider the function

$$zg(z) = \sum_{n=0}^{\infty} c_{n+1} \frac{z^{n+1}}{n!\sqrt{(n+1)!}} = \sum_{m=1}^{\infty} m\sqrt{m+1}c_m\theta_{m+1}(z). \tag{5.10}$$

In order to have $zg(z) \in \mathcal{F}$, it has to verify

$$\sum_{m=1}^{\infty} m^2(m+1)|c_m|^2 < \infty. \tag{5.11}$$

On the other hand, the action of the operator A^+, given in (2.21), on $|g\rangle$ is

$$\begin{aligned} A^+|g\rangle &= b^+a^+b\sum_{m=0}^{\infty} c_{m+1}|\theta_{m+1}\rangle \\ &= \sum_{m=0}^{\infty} c_{m+1}(m+1)\sqrt{m+2}|\theta_{m+2}\rangle = \sum_{n=1}^{\infty} c_n n\sqrt{n+1}|\theta_{n+1}\rangle. \end{aligned} \tag{5.12}$$

It is then clear that A^+ is the operator whose realization in $\mathcal{F}$ is a multiplication by z.

Let us now consider the function

$$\frac{\partial g(z)}{\partial z} = \sum_{m=1}^{\infty} c_{m+1} \frac{z^{m-1}}{(m-1)!\sqrt{(m+1)!}} = \sum_{m=1}^{\infty} \frac{c_{m+1}}{\sqrt{m+1}}\theta_m(z). \tag{5.13}$$

As $[A, A^+] \neq I$, the abstract operator corresponding to the derivative is not A. Therefore, we have to find an operator B such that

$$B|g\rangle = \sum_{m=0}^{\infty} c_{m+1}B|\theta_{m+1}\rangle = \sum_{m=1}^{\infty} \frac{c_{m+1}}{\sqrt{m+1}}|\theta_m\rangle. \tag{5.14}$$

From here, we suppose that it has the form

$$B = b^+af(N)b \qquad N = a^+a \tag{5.15}$$

where the function f is found to be

$$f(N) = \frac{1}{N(1+N)}. \tag{5.16}$$

Indeed, we have

$$\begin{aligned} \frac{1}{\sqrt{m+1}}|\theta_m\rangle &\equiv B|\theta_{m+1}\rangle = b^+af(N)b|\theta_{m+1}\rangle \\ &= \sqrt{m+1}b^+af(N)|\psi_m\rangle = \sqrt{m+1}f(m)b^+a|\psi_m\rangle \\ &= \sqrt{m+1}f(m)\sqrt{m}b^+|\psi_{m-1}\rangle = m\sqrt{m+1}f(m)|\theta_m\rangle \end{aligned} \tag{5.17}$$

then

$$f(N)|\psi_m\rangle = \frac{1}{m(m+1)}|\psi_m\rangle. \tag{5.18}$$

We have then justified the choice of B in (3.3).

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Appendix A. Computation of the measure

In this appendix, we will evaluate $h(x)$, the inverse Mellin transform of the function

$$g(s) = \frac{1}{\pi}(\Gamma(s))^2\Gamma(s+1) = \frac{s}{\pi}(\Gamma(s))^3. \tag{A.1}$$

As is well known [11], a function $h(x)$ and its Mellin transform $g(s)$ are related through

$$g(s) = \int_0^\infty h(x)x^{s-1}\,\mathrm{d}x$$

$$h(x) = \frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} g(s)x^{-s}\,\mathrm{d}s \qquad c \in \mathbb{R}. \tag{A.2}$$

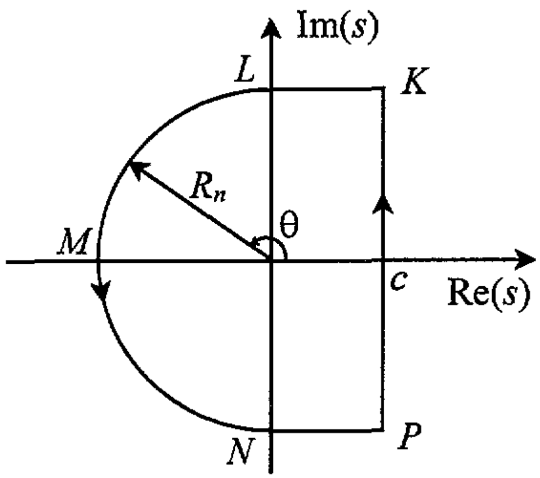

Figure A1. The integration contour for the inverse Mellin transform.

The function $g(s)$, given in (A.1), has poles at $s = 0, -1, -2, \ldots$. To evaluate $h(x)$, let us consider the contour of integration given in figure A1, where

$$R_n = n + \tfrac{1}{2}. \tag{A.3}$$

The function $g(s)$ has no singularity along this contour. Then, according to the residue theorem

$$\frac{1}{2\pi \mathrm{i}}\oint g(s)x^{-s}\,\mathrm{d}s = \sum_{k=0}^{n} \mathrm{Res}[g(s)x^{-s}, s = -k]. \tag{A.4}$$

Taking the limit $n \to \infty$, we have

$$\sum_{k=0}^{\infty} \mathrm{Res}[g(s)x^{-s}, s=-k] = \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} g(s)x^{-s}\,\mathrm{d}s + \frac{1}{2\pi\mathrm{i}} \lim_{n\to\infty} \int_{KLMNP} g(s)x^{-s}\,\mathrm{d}s. \tag{A.5}$$

First, we want to prove that the last integral in (A.5) vanishes. Then, we will compute the sum of the residues.

A1. The integral along the line KLMNP

Let us consider an integral more general than the last one in (A.5)

$$A_n^{k,\ell} = \int_{KLMNP} (\Gamma(s))^k s^\ell x^{-s}\,\mathrm{d}s = \left(\int_{KL} + \int_{LMN} + \int_{NP}\right)(\Gamma(s))^k s^\ell x^{-s}\,\mathrm{d}s \tag{A.6}$$

where $k > 0$. We will show that this integral vanishes in the limit $n \to \infty$ for certain values of k and ℓ. First of all, we have

$$\begin{aligned} |A_n^{k,\ell}| &\leqslant \int_{t=c}^{0} |\Gamma(t+\mathrm{i}R_n)|^k |t+\mathrm{i}R_n|^\ell x^{-t}\,\mathrm{d}t + \int_{\theta=\pi/2}^{3\pi/2} |\Gamma(R_n\mathrm{e}^{\mathrm{i}\theta})|^k R_n^\ell x^{-R_n\cos\theta} R_n\,\mathrm{d}\theta \\ &\quad + \int_{t=0}^{c} |\Gamma(t-\mathrm{i}R_n)|^k |t-\mathrm{i}R_n|^\ell x^{-t}\,\mathrm{d}t \\ &= 2R_n^{\ell+1} \int_{\pi/2}^{\pi} |\Gamma(R_n\mathrm{e}^{\mathrm{i}\theta})|^k \exp(-R_n(\log x)\cos\theta)\,\mathrm{d}\theta. \end{aligned} \tag{A.7}$$

We have to evaluate the last integral in (A.7) for large values of $R_n = n + \frac{1}{2}$. Instead of computing the exact result, we will give the asymptotic value of such an expression. Let us put

$$B(n,k,x,\varepsilon) = \int_{\pi/2}^{\pi-\varepsilon} |\Gamma(R_n\mathrm{e}^{\mathrm{i}\theta})|^k \exp(-R_n(\log x)\cos\theta)\,\mathrm{d}\theta. \tag{A.8}$$

Using Stirling's formula

$$\Gamma(z) \sim \sqrt{2\pi}\, z^{z-(1/2)}\mathrm{e}^{-z} \qquad z \to \infty \qquad |\text{phase } z| \leqslant \pi - \varepsilon < \pi \tag{A.9}$$

we get

$$\begin{aligned} B(n,k,x,\varepsilon) &\sim \int_{\pi/2}^{\pi-\varepsilon} \left|\sqrt{2\pi}(R_n\mathrm{e}^{\mathrm{i}\theta})^{(-1/2)+R_n\exp(\mathrm{i}\theta)} \exp(-R_n\mathrm{e}^{\mathrm{i}\theta})\right|^k \exp(-R_n(\log x)\cos\theta)\,\mathrm{d}\theta \\ &= (2\pi)^{k/2} R_n^{-k/2} \int_{\pi/2}^{\pi-\varepsilon} \exp[R_n\cos\theta(-\log x - k + k\log R_n)] \exp(-kR_n\theta\sin\theta)\,\mathrm{d}\theta \\ &\leqslant (2\pi)^{k/2} R_n^{-k/2} \int_0^{\pi/2} \exp(-u_n\sin\alpha)\,\mathrm{d}\alpha. \end{aligned} \tag{A.10}$$

In the last step, we took into account the fact that the integrand is always positive, as well as the fact that $\exp(-kR_n\theta\sin\theta) \leqslant 1$ in $[(\pi/2), \pi]$. The parameter u_n, which appears in (A.10), is

$$u_n = R_n[k\log R_n - k - \log x]. \tag{A.11}$$

Notice that $u_n > 0$ if n is big enough. In this context, and for $\alpha \in [0, \pi/2]$,

$$-u_n\sin\alpha \leqslant -\frac{2u_n}{\pi}\alpha. \tag{A.12}$$

Therefore, we get

$$B(n,k,x,\varepsilon) \leqslant (2\pi)^{k/2}R_n^{-k/2}\frac{\pi}{2u_n}(1-\mathrm{e}^{-u_n}). \tag{A.13}$$

This result is independent of ε and we then have

$$|A_n^{k,\ell}| \leqslant 2R_n^{\ell+1}\lim_{\varepsilon\to 0}B(n,k,x,\varepsilon) \sim \frac{\pi}{k}(2\pi)^{k/2}\frac{n^{\ell-k/2}}{\log n}. \tag{A.14}$$

We see that $\lim_{n\to\infty}|A_n^{k,\ell}| = 0$ if $\ell \leqslant k/2$. In particular, this is true for $g(s)$, given in (A.1), as $k = 3$, $\ell = 1$. Therefore, we have proved that

$$\frac{1}{2\pi\mathrm{i}}\lim_{n\to\infty}\int_{KLMNP} g(s)x^{-s}\,\mathrm{d}s = 0 \tag{A.15}$$

and consequently

$$\pi h(x) = \sum_{n=0}^{\infty}\mathrm{Res}[s(\Gamma(s))^3x^{-s}, s=-n]. \tag{A.16}$$

A2. The sum of the residues

Let us calculate the residues that appear in (A.16). Remembering one of the definitions of the gamma function

$$\Gamma(z) = \frac{1}{z}\prod_{k=1}^{\infty}\frac{\left(1+\frac{1}{k}\right)^z}{1+\frac{z}{k}} \tag{A.17}$$

it is clear that the function $x^{-s}s(\Gamma(s))^3$ has a double pole at $s = 0$ and triple poles at $s = -1, -2, -3, \ldots$. Let us first evaluate

$$\begin{aligned}\mathrm{Res}\left[s(\Gamma(s))^3x^{-s}, s=0\right] &= \lim_{s\to 0}\frac{\mathrm{d}}{\mathrm{d}s}\left[(s\Gamma(s))^3x^{-s}\right]\\ &= \lim_{s\to 0}\left[-x^{-s}(\log x)(\Gamma(s+1))^3 + 3(\Gamma(s+1))^2\frac{\mathrm{d}\,\Gamma(s+1)}{\mathrm{d}s}\right]\\ &= -\log x + 3\lim_{s\to 0}\psi(s+1) = -(3\gamma + \log x).\end{aligned} \tag{A.18}$$

Let us now consider the triple poles

$$\mathrm{Res}\left[s(\Gamma(s))^3x^{-s}, s=-n\neq 0\right] = \frac{1}{2}\lim_{s\to -n}\frac{\mathrm{d}^2}{\mathrm{d}s^2}\left[((s+n)\Gamma(s))^3 s x^{-s}\right]$$
$$= \frac{1}{2}x^n\Bigg[\left(-2\log x - n(\log x)^2\right)\lim_{s\to -n} a(s)^3 + (6+6n\log x)\lim_{s\to -n}\left[a(s)^2\frac{\mathrm{d}a(s)}{\mathrm{d}s}\right]$$
$$- 6n\lim_{s\to -n} a(s)\left(\frac{\mathrm{d}a(s)}{\mathrm{d}s}\right)^2 - 3n\lim_{s\to -n}\left(a(s)^2\frac{\mathrm{d}^2a(s)}{\mathrm{d}s^2}\right)\Bigg] \qquad \text{(A.19)}$$

where $a(s) = (s+n)\Gamma(s)$. Then, we have

$$\lim_{s\to -n} a(s) = \lim_{s\to -n}(s+n)\Gamma(s) = \mathrm{Res}[\Gamma(s), s=-n] = \frac{(-1)^n}{n!}$$
$$\lim_{s\to -n}\frac{\mathrm{d}a(s)}{\mathrm{d}s} = \lim_{s\to -n}\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\Gamma(s+n+1)}{(s+n-1)\cdots(s+1)s}\right]$$
$$= \lim_{s\to -n}\frac{\Gamma(s+n+1)}{(s+n-1)\cdots s}\frac{\mathrm{d}}{\mathrm{d}s}\left[\log\Gamma(s+n+1) - \sum_{k=1}^{n}\log(s+n-k)\right]$$
$$= \lim_{s\to -n} a(s)\left[\psi(s+n+1) - \sum_{k=1}^{n}\frac{1}{s+n-k}\right] = \frac{(-1)^n}{n!}\left[-\gamma + \sum_{k=1}^{n}\frac{1}{k}\right]$$
$$\lim_{s\to -n}\frac{\mathrm{d}^2a(s)}{\mathrm{d}s^2} = \lim_{s\to -n} a(s)\Bigg[\left(\psi(s+n+1) - \sum_{k=1}^{n}\frac{1}{s+n-k}\right)^2$$
$$+ \psi'(s+n+1) + \sum_{k=1}^{n}\frac{1}{(s+n-k)^2}\Bigg]$$
$$= \frac{(-1)^n}{n!}\left[\left(-\gamma + \sum_{k=1}^{n}\frac{1}{k}\right)^2 + \frac{\pi^2}{6} + \sum_{k=1}^{n}\frac{1}{k^2}\right]$$

where we have used well known properties of the gamma and psi functions [12]. The residue (A.19) is

$$\mathrm{Res}\left[s(\Gamma(s))^3x^{-s}, s=-n\right] = \frac{(-x)^n}{2(n!)^3}\Bigg[-n(\log x)^2 + \left(-2 + 6n\left(\sum_{k=1}^{n}(1/k) - \gamma\right)\right)\log x$$
$$+ 6\left(\sum_{k=1}^{n}(1/k) - \gamma\right) - 3n\left(\frac{\pi^2}{6} + \sum_{k=1}^{n}(1/k^2) + 3\left(\sum_{k=1}^{n}(1/k) - \gamma\right)^2\right)\Bigg]. \qquad \text{(A.20)}$$

We can finally write the function $h(x)$ in (A.16):

$$h(x) = -\frac{3\gamma}{\pi} + \frac{x(\log x)^2}{2\pi}{}_0F_2(2,2;-x) - \frac{\log x}{\pi}{}_0F_2(1,1;-x)$$
$$- \frac{3x\log x}{\pi}\sum_{n=0}^{\infty}\gamma_{n+1}\frac{(-x)^n}{((n+1)!)^2n!} - \frac{3x}{\pi}\sum_{n=0}^{\infty}\gamma_{n+1}\frac{(-x)^n}{((n+1)!)^3}$$
$$+ \frac{9x}{2\pi}\sum_{n=0}^{\infty}\gamma_{n+1}^2\frac{(-x)^n}{((n+1)!)^2n!} + \frac{3x}{2\pi}\sum_{n=0}^{\infty}\left(\frac{\pi^2}{6} + \sum_{k=1}^{n+1}\frac{1}{k^2}\right)\frac{(-x)^n}{((n+1)!)^2n!} \qquad \text{(A.21)}$$

where

$$\gamma_{n+1} = \sum_{k=1}^{n+1} \frac{1}{k} - \gamma \tag{A.22}$$

and where γ is Euler's constant ($\gamma = 0.5772\ldots$). Due to the definition of Euler's constant, the asymptotic behaviour of γ_{n+1} for large values of n is $\gamma_{n+1} \sim \log(1+n)$. In addition, $\sum_{k=1}^{\infty}(1/k)^2 = \pi^2/6$. Therefore, the form of $h(x)$ is essentially a sum of functions ${}_0F_2(a, b; -x)$ multiplied by x and/or powers of $\log x$. Notice that $h(x)$ is singular at $x = 0$, indeed,

$$h(x) \underset{x\to 0}{\approx} -\frac{1}{\pi} \log x. \tag{A.23}$$

Nevertheless, this is not a problem because the important objects are the measures $\mathrm{d}\mu(z)$ and $\mathrm{d}\sigma(z)$, given by

$$\mathrm{d}\mu(z) = {}_0F_2(1, 2; r^2)h(r^2)r\,\mathrm{d}r\,\mathrm{d}\varphi = {}_0F_2(1, 2; r^2)\,\mathrm{d}\sigma(z) \tag{A.24}$$

which are well behaved for $r \to 0$.

Appendix B. Order and type of the functions in $\mathcal{F}$

The order and type of an entire function [13] give us information on the growth of the function at infinity. Let us consider an entire function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{B.1}$$

The exponential function is used to measure the growth of $f(z)$ or, more precisely, the growth of $|f(z)|$. Suppose that there exist positive numbers μ and k such that

$$\max_{|z|=r} |f(z)| < \exp(kr^{\mu}) \tag{B.2}$$

for all sufficiently large r. The greatest lower bound of the numbers μ verifying (B.2) is called the order ϱ of $f(z)$; the greatest lower bound of the numbers k for which (B.2) holds is called the type of $f(z)$ and is usually denoted by σ. Both ϱ and σ are greater than or equal to zero.

The order and type of an entire function depend on the coefficients of its Taylor series expansion (B.1). Indeed, it can be proved that if $f(z)$ is an entire function of order ϱ, then

$$\varrho = \overline{\lim_{n\to\infty}} \left(\frac{-n \log n}{\log |a_n|} \right). \tag{B.3}$$

In addition, if $f(z)$ is an entire function of finite order ϱ and type σ, then

$$\sigma = \frac{1}{\mathrm{e}\varrho} \overline{\lim_{n\to\infty}} \left(n |a_n|^{\varrho/n} \right). \tag{B.4}$$

We are particularly involved with generalized hypergeometric functions

$$
\begin{aligned}
{}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z) &\equiv {}_pF_q(a_i;b_j;z)\\
&= \frac{\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(a_1)\ldots\Gamma(a_p)}\sum_{n=0}^{\infty}\frac{\Gamma(a_1+n)\ldots\Gamma(a_p+n)}{\Gamma(b_1+n)\ldots\Gamma(b_q+n)}\frac{z^n}{n!}.
\end{aligned}
\tag{B.5}
$$

Comparing this with (B.1), we have in this case

$$
a_n = k\frac{\Gamma(a_1+n)\ldots\Gamma(a_p+n)}{\Gamma(b_1+n)\ldots\Gamma(b_q+n)}\frac{1}{n!} \qquad k=\frac{\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(a_1)\ldots\Gamma(a_p)}.
\tag{B.6}
$$

Let us compute the order and type of such functions. Using (B.3) with (B.6), and taking into account Stirling's formula, we get

$$
\varrho = \overline{\lim_{n\to\infty}}\left(\frac{-n\log n}{\log|k|+(1+q-p)n(1-\log n)}\right)=\frac{1}{1+q-p}.
\tag{B.7}
$$

By definition $\varrho \geqslant 0$, then $p \leqslant q+1$. A similar analysis shows that, for ${}_pF_q(a_i;b_j;z^s)$, the order is $\varrho_s = s/(1+q-p)$.

Let us consider now the type of ${}_pF_q(a_i;b_j;z)$:

$$
\begin{aligned}
\sigma &= \frac{1}{e\varrho}\exp\left[\overline{\lim_{n\to\infty}}\left(\log n+\frac{\varrho}{n}[\log|k|+n(1-\log n)(1+q-p)]\right)\right]\\
&= \frac{1}{\varrho} = 1+q-p.
\end{aligned}
\tag{B.8}
$$

For the function ${}_pF_q(a_i;b_j;z^s)$, we get $\sigma_s = s/\varrho_s = 1+q-p$. In the particular case of ${}_0F_2(1,2;|z|^2)$, we find $\varrho=\frac{2}{3}$ and $\sigma=3$. Then the function $\sqrt{{}_0F_2(1,2;|z|^2)}$ has $\varrho=\frac{2}{3}$ and $\sigma=\frac{3}{2}$ (because if $|f(z)|\leqslant A\exp(\sigma r^{\varrho})$ then $|\sqrt{f(z)}|=\sqrt{|f(x)|}\leqslant\sqrt{A}\exp((\sigma/2)r^{\varrho})$).

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